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THE STABILITY OF RELATIVE EQUILIBRIA OF AN ELASTIC SATELLITE WITH RESPECT TO SOME OF THE VARIABLES†

L. YU. ANAPOL'SKII and S. V. CHAIKIN

Irkutsk

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The restricted formulation [1] of the problem of the stability of the motion of an elastic satellite around a circular orbit in a Newtonian central field of force is considered. The satellite is modelled by a rigid body with an attached arbitrary isotropic elastic element. It is assumed that the vector of a small deformation of the satellite can be represented as an infinite series in terms of the known orthogonal characteristic modes of free oscillations of the satellite [2, 3]. Using the theorem in [4] on the stability with respect to some of the variables, one can specify sufficient conditions for the stability of relative equilibrium states of the satellite with respect to the velocities and position coordinates, including an arbitrary finite number of generalized coordinates defining the deformed state of the elastic satellite. The influence of the discarded elastic coordinates (which define the elastic deformation) is taken into account. Equations for finding the deformations of the satellite in a relative equilibrium state are presented, the state being always characterized by the fact that the main central axes constructed for the given equilibrium state of the satellite are parallel to the axes of the orbital coordinate system. Necessary and sufficient conditions for the lack of deformation of the satellite in a relative equilibrium state are given.

1. WITHIN a restricted formulation [1] we consider the motion of an elastic satellite in a Newtonian central field of force, the satellite being modelled by a rigid body with an elastic element attached to it. The centre of mass of the satellite moves at a constant angular velocity around a Kepler circular orbit of radius R_0 with an attracting centre.

We introduce the following right-handed orthogonal Cartesian systems of coordinates: $Oy_1y_2y_3$ is the orbital coordinate system (OCS) with origin at the centre of mass of the satellite and unit vectors $\boldsymbol{\beta}, \boldsymbol{\gamma}$ of the axes Oy_2, Oy_3, the axis Oy_3 is parallel to the radius vector of the point O relative to the attracting centre, the axis Oy_1 is parallel to the transversal direction to the orbit at O and points in the direction of motion of the satellite, $\boldsymbol{\omega} = |\omega|\boldsymbol{\beta}$ denotes the angular velocity of orbital motion with $\omega = \text{const} > 0$, $Ox_1x_2x_3$ is the coordinate system rigidly attached to the satellite (SCS) with unit vectors $\mathbf{i}^k$ $(k = 1, 2, 3)$ of the axes, with origin at the centre of mass O_1 of the non-deformed satellite, and with axes parallel to the main central axes of the satellite, $Ox_1x_2x_3$ is the coordinate system with origin at the centre of mass O of the satellite and unit vectors $\mathbf{i}^k$ of the axes, respectively, and Ω denotes the angular velocity of the trihedron $Ox_1x_2x_3$ relative to $Oy_1y_2y_3$.

The positions of relative equilibrium of the elastic satellite are those in which the satellite is at rest relative to the OCS. If, in a position of relative equilibrium the elastic element of the satellite is in a deformed state, then the equilibrium will be called non-trivial.

Suppose that the points of the rigid body of the satellite occupy a bounded region υ_1 and the

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points of the elastic element occupy a bounded region υ_2 in the non-deformed state. Let Γ be the common boundary of the regions υ_1 and υ_2, and let $\upsilon \equiv \upsilon_1 + \upsilon_2$. We shall assume that the regions are defined relative to the SCS.

We denote by $\mathbf{r}$ ($\mathbf{r} \in \upsilon$) the radius-vector of an arbitrary point M of the satellite. When there is a deformation $\boldsymbol{\rho} = \mathbf{r} + \mathbf{u}(t, \mathbf{r})$ will be the radius-vector of M relative to O. Here $t \in [0, \infty)$ is the time and the function $\mathbf{u}$: $(t, \mathbf{r}) \to \mathbf{u}(t, \mathbf{r})$ is sufficiently smooth in t, and $\mathbf{r}$ and defines the displacement of M resulting from a small deformation of the elastic element.

Let us state the following assumptions to be used later on.

1. The displacement vector $\mathbf{u}(t, \mathbf{r})$ can be represented as an infinite series in terms of the characteristic modes of the whole satellite

$$\mathbf{u}(t,\mathbf{r}) = \Sigma q_n(t)\boldsymbol{\Psi}_n(\mathbf{r}) \tag{1.1}$$

where $\boldsymbol{\Psi}_n(r) = (\Psi_n^1,\ \Psi_n^2,\ \Psi_n^3)$ are the characteristic modes of elastic oscillations of the satellite relative to the non-deformed state with components expressed in the system $Ox_1x_2x_3$, and where $q_n(t)$ is the generalized coordinate of the nth mode of elastic oscillations. We assume that the system $\{\boldsymbol{\Psi}_n\}$ is orthogonal ($\int_\upsilon \Psi_n^i \Psi_p^i dm = 0$ if $i \neq j$ or $n \neq p$ [2, 3]) and $\mathbf{q} = \{q_n\} \in l_2$ is the Hilbert space of sequences with bounded norm $\|\mathbf{q}\| = (\Sigma q_n^2)^{1/2}$. Here and in what follows, unless otherwise stipulated, summation is carried out over n from 1 to ∞. We assume that the conditions for the convergence of the functional series are such that all the operations to be applied to the series in the present paper are correct.

2. The central ellipsoid of inertia of the satellite is triaxial in the non-deformed case.

3. The potential energy of gravitational forces is given by the approximate expression [1]

$$\Pi_g = -\mu m / R_0 + \omega^2(3\boldsymbol{\gamma}\mathbf{J}\boldsymbol{\gamma} - \operatorname{tr}\mathbf{J})/2 \tag{1.2}$$

where m is the mass of the satellite, μ is the product of the universal constant of gravitation and the mass of the attracting centre, and $\mathbf{J}$ is the inertia tensor of the satellite about the centre of mass O.

4. The elastic element of the satellite is assumed to be isotropic. In the case of small deformations its potential energy can be represented as (cf. [2])

$$\Pi = \frac{1}{2}\Sigma c_{nn} q_n^2 \tag{1.3}$$

where $c_{nn} > 0$ are real constants.

5. In what follows we shall consider only those $\mathbf{q} = \{q_n\}$ for which the kinetic energy of the satellite about its centre of mass is a positive definite form of Ω and $q^\bullet = \{q^\bullet_n\}$, $({}^\bullet = \partial/\partial t)$.

The Greek letters β, γ, ω, Ω, ρ, Ψ denote the corresponding vectors, the components of which are denoted by indices.

Taking (1.1) into account, we represent the inertia tensor in the form [2–4]

$$\mathbf{J} = \int_\upsilon (\boldsymbol{\rho}\cdot\boldsymbol{\rho}E - \boldsymbol{\rho}{:}\boldsymbol{\rho})dm = \mathbf{J}_0 + \Sigma q_n \mathbf{J}_n + \Sigma q_n^2 \mathbf{J}_{nn} \tag{1.4}$$

Here and henceforth $\mathbf{a}\cdot\mathbf{b}$ is the scalar product of $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a}:\mathbf{b}$ is the dyadic product of $\mathbf{a}$ and $\mathbf{b}$, E is the unit matrix, and integration is carried out over υ.

The matrices formed by the components of $\mathbf{J}$, $\mathbf{J}_0$, $\mathbf{J}_n$, $\mathbf{J}_{nn}$ in the coordinate system $Ox_1x_2x_3$ read

$$I = \|I^{ij}\| = I_0 + \Sigma q_n I_n + \Sigma q_n^2 I_{nn}$$

$$I_0 = \operatorname{diag}(\mu^1, \mu^2, \mu^3), \quad I_n = \|I_n^{ij}\|$$

$$I_n^{ij} = -H_n^{ji} = -\int (x_i \Psi_n^j + x_j \Psi_n^i)dm, \; I_n^{ii} = -\sum_{j=1}^{3} H_n^{jj} + H_n^{ii}$$

$$I_{nn} = \| I_{nn}^{ij} \| = \operatorname{diag}(L_{nn}^{22} + L_{nn}^{33}, L_{nn}^{11} + L_{nn}^{33}, L_{nn}^{11} + L_{nn}^{22})$$
$$L_{nn}^{ii} = \int \Psi_n^i \Psi_n^i dm (i, j = 1,2,3; n = 1,2,\ldots)$$

where μ^k ($k = 1, 2, 3$) are the main central moments of inertia of the non-deformed satellite, and x_j is the jth component of $\mathbf{r}$ in the SCS [2].

Various approaches to the construction of the equations of motion of complex mechanical systems were discussed in [5, 6]. In the case under consideration, the equations of motion can be obtained from the equations in [5, p. 45].

In the present paper the form of the equations of motion will neither be used nor stated.

2. We know [1–6] that, in the case under consideration, along with the particular integrals

$$U_1 \equiv \boldsymbol{\gamma\gamma} - 1 = 0, U_2 \equiv \boldsymbol{\beta\beta} - 1 = 0, U_3 \equiv \boldsymbol{\gamma\beta} = 0 \tag{2.1}$$

of the direction cosines, the equations of motion of an elastic satellite relative to the centre of mass admit of the Jacobi type integral

$$U \equiv T_0 + \Pi + \Pi_g - \boldsymbol{\omega} \mathbf{J} \boldsymbol{\omega} / 2 = \text{const} \tag{2.2}$$

where $T_0 = \boldsymbol{\Omega}\mathbf{J}\boldsymbol{\Omega}/2 + \boldsymbol{\Omega}\mathbf{G} + T$ is the kinetic energy of the satellite relative to O, $\mathbf{G}$ and T being the vector of the kinetic momentum of the satellite relative to O and its kinetic energy due to the elastic deformations only.

Using representation (1.1), we have

$$\mathbf{G} \equiv \int (\mathbf{r} + \mathbf{u}) \times \mathbf{u}^{\cdot} dm = \sum_n \mathbf{G}_n q_n^{\cdot} + \sum_{n,m} \mathbf{G}_{mn} q_m q_n^{\cdot}$$
$$\mathbf{G}_n \equiv \int \mathbf{r} \times \boldsymbol{\Psi}_n dm, \mathbf{G}_{mn} \equiv \int \boldsymbol{\Psi}_m \times \boldsymbol{\Psi}_n dm \tag{2.3}$$

$$T \equiv \frac{1}{2} \int \mathbf{u}^{\cdot} \mathbf{u}^{\cdot} dm = \frac{1}{2} \sum a_{nn} q_n^{\cdot} q_n^{\cdot}, a_{nn} > 0 \tag{2.4}$$

Suppose that the values

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}^0, \boldsymbol{\beta} = \boldsymbol{\beta}^0, \boldsymbol{\gamma} = \boldsymbol{\gamma}^0, \mathbf{q} = \mathbf{q}^0, \mathbf{q}^{\cdot} = \mathbf{q}^{0\cdot} \tag{2.5}$$

of the variables of the problem define the equilibrium state of the satellite (unperturbed motion) and

$$\boldsymbol{\Omega}^* = \boldsymbol{\Omega} - \boldsymbol{\Omega}^0, \boldsymbol{\beta}^* = \boldsymbol{\beta} - \boldsymbol{\beta}^0, \boldsymbol{\gamma}^* = \boldsymbol{\gamma} - \boldsymbol{\gamma}^0, \tag{2.6}$$
$$\mathbf{q}^{*\cdot} = \mathbf{q}^{\cdot} - \mathbf{q}^{0\cdot}, \mathbf{q}^* = \mathbf{q} - \mathbf{q}^0$$

describe the deviation from the unperturbed motion.

We introduce the functions

$$V(\boldsymbol{\Omega}^*, \boldsymbol{\gamma}^*, \boldsymbol{\beta}^*, \mathbf{q}^*, \mathbf{q}^{*\cdot}) = U + 3\omega^2 \lambda(\mathbf{q}^*) U_3 - 3\omega^2 \sigma(\mathbf{q}^*) U_1 / 2 + \omega^2 v(\mathbf{q}^*) U_2$$
$$V_1(\mathbf{q}^*, \boldsymbol{\gamma}^*, \boldsymbol{\beta}^*) = \Pi + \Pi_g - \boldsymbol{\omega}\mathbf{J}\boldsymbol{\omega}/2 + 3\omega^2 \lambda(\mathbf{q}^*) U_3 - 3\omega^2 \sigma(\mathbf{q}^*) U_1 + \omega^2 v(\mathbf{q}^*) U_2 \tag{2.7}$$
$$V(\boldsymbol{\Omega}^*, \boldsymbol{\gamma}^*, \boldsymbol{\beta}^*, \mathbf{q}^*, \mathbf{q}^{*\cdot}) = T_0 + V_1$$

where the undefined Lagrange multipliers can be represented in the form

$$\lambda(q^*)=\lambda_0+\sum\lambda_n(q_n^*+q_n^0)+\sum\lambda_{nn}(q_n^*+q_n^0)^2$$
$$\sigma(q^*)=\sigma_0+\sum\sigma_n q_n^*+\sum\sigma_{nn}(q_n^*)^2 \tag{2.8}$$
$$\nu(q^*)=\nu_0+\sum\nu_n q_n^*+\sum\nu_{nn}(q_n^*)^2$$

By the equations of the unperturbed motion and Poisson's kinematic equations

$$\boldsymbol{\beta}^{\cdot}=\boldsymbol{\beta}\times\boldsymbol{\Omega},\quad \boldsymbol{\gamma}^{\cdot}=\boldsymbol{\gamma}\times\boldsymbol{\Omega}$$

we have $dV/dt=0$.

According to Routh's theorem the equations for finding the quantities (2.5) and the undetermined Lagrange multipliers (2.8) obtained from the condition $\delta V=0$ for $\boldsymbol{\Omega}^*=0,\ldots,$ $\mathbf{q}^{*\cdot}=0$ can be written as follows:

$$\boldsymbol{\gamma}^0\cdot\boldsymbol{\gamma}^0-1=0,\quad \boldsymbol{\beta}^0\cdot\boldsymbol{\beta}^0-1=0,\quad \boldsymbol{\gamma}^0\cdot\boldsymbol{\beta}^0=0 \tag{2.9}$$

$$3\omega^2((\mathbf{J}(q^0)-\sigma(0)E)\boldsymbol{\gamma}^0+\lambda(0)\boldsymbol{\beta}^0)=0$$
$$\omega^2((\nu(0)E-\mathbf{J}(q^0))\boldsymbol{\beta}^0+3\lambda(0)\boldsymbol{\gamma}^0)=0 \tag{2.10}$$

$$\boldsymbol{\Omega}^0(\mathbf{J}_n+2q_n\mathbf{J}_{nn})\boldsymbol{\Omega}^0/2+\boldsymbol{\Omega}^0(\sum_m\mathbf{G}_{mn}q_m^{0\cdot})+c_{nn}q_n^0+$$
$$+3\omega^2\boldsymbol{\gamma}^0(\mathbf{J}_n+2q_n^0\mathbf{J}_{nn})\boldsymbol{\gamma}^0/2-\omega^2\operatorname{tr}(\mathbf{J}_n+2q_n^0\mathbf{J}_{nn})/2-$$
$$-\omega^2\boldsymbol{\beta}^0(\mathbf{J}_n+2q_n^0\mathbf{J}_{nn})\boldsymbol{\beta}^0/2=0 \tag{2.11}$$
$$\mathbf{J}(q^0)\boldsymbol{\Omega}^0+\sum_m(\mathbf{G}_m q_m^{0\cdot}+\sum_p\mathbf{G}_{pm}q_p^0)q_m^{0\cdot}=0$$

$$(\sum_m\mathbf{G}_{mn}q_m^0+\mathbf{G}_n)\boldsymbol{\Omega}+a_{nn}q_n^{0\cdot}=0 \tag{2.12}$$

where $\mathbf{J}(q^0)$ is the value of (1.4) for $q=q^0$.

Solving the second group of equations (2.12) for $q_n^{0\cdot}$ $(a_{nn}>0)$ and substituting the resulting values into the first (vector) equation (2.12), we arrive at the conclusion that system (2.12) has a unique solution $\boldsymbol{\Omega}^0=0$, $\mathbf{q}^{0\cdot}=0$, subject to the condition

$$\det(I(q^0)+\sum G_n^0 G_n^{0T}a_{nn}^{-1})\neq 0\quad \mathbf{G}_n^0=\mathbf{G}_n+\sum_m\mathbf{G}_{mn}q_m^0 \tag{2.13}$$

where G_n^0 is the column formed by the components of $\mathbf{G}_n^0$ in the coordinate system $Ox_1x_2x_3$. The violation of (2.13) contradicts assumption 5.

To solve Eqs (2.9)–(2.11) and carry out the subsequent investigation, we use the coordinate system $Ox_1^0x_2^0x_3^0$ with origin at the centre of mass of the satellite, in which the matrix formed by the components of the tensor of inertia $\mathbf{J}(\mathbf{q}^0)$ is diagonal. We denote by $\mathbf{e}^k(\mathbf{q}^0)$ the unit vectors of the corresponding axes and by $P(\mathbf{q}^0)$ the orthogonal matrix of the transformation of the system $Ox_1x_2x_3$ to $Ox_1^0x_2^0x_3^0$.

The same notation will be used for the matrix of components of any of the tensors encountered in this paper relative to the coordinate system $Ox_1^0x_2^0x_3^0$.

In view of (2.9), taking the scalar product of the first equation in (2.10) and $\boldsymbol{\beta}^0$ as well as the scalar product of the second equation and $\boldsymbol{\gamma}^0$, we obtain $\lambda(0)=-\beta^{0T}J(q^0)\gamma^0$ and $3\lambda^0=\gamma^{0T}J(q^0)\beta^0$, which implies that $\lambda(0)=0$ independently of the values q_n^0 defining the unperturbed motion. It follows that $\lambda(\mathbf{q}^*)\equiv 0$, that is, $\lambda_0=0$, $\lambda_n=0$, $\lambda_{nn}=0$, γ^0 and β^0 being distinct unit eigenvectors of $\mathbf{J}(\mathbf{q}^0)$ with eigenvalues σ_0, ν_0 and components $\beta^0=(\beta_1^0,\ \beta_2^0,\ \beta_3^0)^T$, $\gamma^0=(\gamma_1^0,\ \gamma_2^0,\ \gamma_3^0)^T$.

By the method of perturbation theory [7], we can determine the eigenvalues $\mu^k(\mathbf{q}^0)$ and the corresponding orthonormal eigenvectors $\mathbf{e}^k(\mathbf{q}^0)$ of the matrix $I(\mathbf{q}^0)$ as power series in q_n^0

$$\mu^k(q^0) = I_0^k + \Sigma q_n^0 \mu_n^k + \ldots, \quad \mu_n^k = I_n^{kk}$$
$$\mathbf{e}^k(q^0) = \mathbf{i}^k + \Sigma q_n^0 \mathbf{i}_n^k + \ldots, \quad \mathbf{i}_n^k = \sum_{j=1, j\neq k}^{3} (I_0^k - I_0^j)^{-1} I_n^{jk} \mathbf{i}^j \tag{2.14}$$

$(I_0^k, \mathbf{i}^k)$ being a characteristic pair of the matrix $I_0 = \operatorname{diag}(I_0^1, I_0^2, I_0^3)$.

It is obvious that, using (2.14), the orthogonal transformation matrix $P(q^0)$ can be represented as

$$P(\mathbf{q}^0) \equiv (e^1(\mathbf{q}^0), e^2(\mathbf{q}^0), e^3(\mathbf{q}^0)) = E + \Sigma q_n^0 P_n + \ldots \tag{2.15}$$

where $e^k(\mathbf{q}^0)$ is the column formed by the components of $e^k(\mathbf{q}^0)$ in the system $Ox_1x_2x_3$.

By the theorem on the transformation of the components of a tensor when changing to a new system of coordinates, we have

$$J_n \equiv \left\| J_n^{ij} \right\| = P(\mathbf{q}^0) I_n P^t(\mathbf{q}^0), \quad J_{nn} \equiv \left\| J_{nn}^{ij} \right\| = P(\mathbf{q}^0) I_{nn} P^T(\mathbf{q}^0) \tag{2.16}$$

$$\begin{aligned}
&J(\mathbf{q}^*) = P(\mathbf{q}^*)(I_0 + \Sigma q_n^0 I_n + \Sigma q_n^{02} I_{nn} + \\
&+ \Sigma q_n^* (I_n + 2q_n^0 I_{nn}) + \Sigma q_n^{*2} I_{nn}) P^t(\mathbf{q}^0) = J_0 + \\
&+ \Sigma q_n^* (J_n + 2q_n^0 J_{nn}) + \Sigma q_n^{*2} J_{nn} \\
&J_0 = P(\mathbf{q}^0)(I_0 + \Sigma q_n^0 I_n + \Sigma q_n^{02} I_{nn}) P^t(\mathbf{q}^0) = \\
&= \operatorname{diag}(J_0^1(\mathbf{q}^0), J_0^2(\mathbf{q}^0), J_0^3(\mathbf{q}^0)), \quad J_0^k = \mu^k(\mathbf{q}^0)
\end{aligned}$$

In terms of the projections onto the axes $Ox_1^0x_2^0x_3^0$, the solution of Eqs (2.9) and (2.10) can be written as follows:

$$\begin{aligned}
&\forall k, l, m \in \{1,2,3\}, \ k \neq l \neq m \\
&\Omega_i^0 = 0, \ \mathbf{q}^{0\cdot} = 0, \ \lambda_0 = 0, \ \lambda_n = 0, \ \lambda_{nn} = 0 \ (n = 1,2,\ldots) \\
&\nu_0 = \mu^k(\mathbf{q}^0), \ \beta_i^0 = \pm\delta_{ik} \\
&\sigma_0 = \mu^m(\mathbf{q}^0), \ \gamma_i^0 = \pm\delta_{im} (i = 1,2,3)
\end{aligned} \tag{2.17}$$

where δ_{ij} is the Kronecker delta, and where q_n^0 can be determined from the following equations obtained from (2.11) taking (2.17) into account

$$\begin{aligned}
&c_{nn} q_n^0 + 3\omega^2 (J_n^{mm} + 2q_n^0 J_{nn}^{mm})/2 - \omega^2 \operatorname{tr}(J_n + 2q_n^0 J_{nn})/2 - \\
&-\omega^2 (J_n^{kk} + 2q_n^0 J_{nn}^{kk})/2 = 0 \ (n = 1.2,\ldots)
\end{aligned} \tag{2.18}$$

The quantities σ_n, ν_n, σ_{nn}, ν_{nn} from (2.8) can be determined by investigating the conditions for the solutions of (2.17) and (2.18) to be stable.

If system (2.18) has no real solutions, the satellite has no positions of relative equilibrium. The assertion below follows from (2.17).

Assertion 1. For a fixed system of integers k, l, $m \in \{1, 2, 3\}$, every solution of (2.18) defines four distinct equilibrium states of the satellite

$$\Omega_i^0 = 0,\ \mathbf{q}^{0\cdot} = 0$$

$$\beta_i^0(\mathbf{q}^0) = \pm\delta_{ik},\ \gamma_i^0(\mathbf{q}^0) = \pm\delta_{im},\ i = 1,2,3 \tag{2.19}$$

If a position of relative equilibrium of the satellite exists, its main central axes constructed for a given equilibrium state are parallel to the axes of the orbital coordinate system. The assertion below follows if formulae (2.15) and (2.16) are taken into account.

Assertion 2. In order that a trivial relative equilibrium state defined by $q_n^0 = 0$ exists, it is necessary and sufficient that

$$I_n^{kk} - I_n^{mm} + I_n^{ll}/2 = 0 \tag{2.20}$$

for each $n = 1, 2, \ldots$.

Equations (2.20) impose certain restrictions on the position and characteristics of the elastic element.

3. Suppose that Eqs (2.18) have an isolated solution and (2.19) are the positions of equilibrium of the satellite (unperturbed motions) corresponding to the solution. We will use Theorem 1 to study the stability of these states with respect to some of the variables [4, p. 131].

As the Lyapunov function we take

$$W(\Omega^*, \gamma^*, \beta^*, q^*, q^{*\cdot}) \equiv V(\Omega^*, \gamma^*, \beta^*, \mathbf{q}^*, \mathbf{q}^{*\cdot}, \sigma_0, \nu_0) - V(0)$$

where $V(0)$ is the value of V for the unperturbed motion (see (2.7)) and $dW/dt = 0$ by virtue of the equations of perturbed motion.

In a small neighbourhood of the unperturbed motion

$$W = T_0 + V_1 - V_1(0) = T_0 + \delta V_1(0) + \delta^2 V_1(0) + \ldots \tag{3.1}$$

If assumption 5 and the fact that $\delta V_1(0) = 0$ are taken into account, then the conditions ensuring that $\delta^2 V_1(0)$ is positive definite with respect to some of the variables

$$\delta w_1 = \delta\gamma_1^*,\quad \delta w_2 = \delta\beta_1^*,\quad \delta w_3 = \delta\gamma_2^*,\quad \delta w_4 = \delta\beta_2^*$$

$$\delta w_5 = \delta\gamma_3^*,\quad \delta w_6 = \delta\beta_3^*,\quad \delta w_{6+i} = \delta q_i^*\ (i = 1, \ldots, N)$$

on the linear manifolds defined by the equalities $\delta U_j = 0$ $(j = 1, 2, 3)$ will be sufficient in order for the positions of equilibrium of the elastic satellite with respect to Ω_j, β_j, γ_i, $q_1, \ldots q_N$, $q_1^\cdot$, $q_2^\cdot$ to be stable.

We represent $\delta^2 V_1(0)$ as a denumerably-dimensional quadratic form with real matrix H_1

$$\delta^2 V_1(0) = \delta w^T H_1 \delta \mathbf{w},\ H_1 = \begin{Vmatrix} A_1 & B \\ B & C \end{Vmatrix} \tag{3.2}$$

$$\mathbf{w} \equiv (w_1, w_2, \ldots)^T,\ B^T = (b_1, \ldots, b_6),\ b_i = (b_{i1}, b_{i2}, \ldots)^T$$

$$\begin{Vmatrix} b_{1n} \\ b_{3n} \\ b_{5n} \end{Vmatrix} = 3((J_n - \sigma_n E) + 2q_n^0(J_{nn} - \sigma_{nn} E))\gamma_0 \tag{3.3}$$

$$\begin{Vmatrix} b_{2n} \\ b_{4n} \\ b_{6n} \end{Vmatrix} = ((\nu_n E - J_n) + 2q_n^0(\nu_{nn}E - J_{nn}))\beta^0$$

$$i = 1,2,\ldots,6; \quad n = 1,2,\ldots$$

The diagonal matrix

$$\begin{aligned} C &= \left\| \omega^{-2} c_{nn} + 2(J_{nn}^{mm} - J_{nn}^{kk} - J_{nn}^{11}/2) \right\|, n = 1,2,\ldots \\ A_1 &= \mathrm{diag}(3(J_0^1 - \sigma_0), (\nu_0 - J_0^1), 3(J_0^2 - \sigma_0), (\nu_0 - J_0^2), 3(J_0^3 - \sigma_0), (\nu_0 - J_0^3)) \end{aligned} \tag{3.4}$$

Remark. For any solution (2.17), (2.18), two of the diagonal elements of A_1 vanish and the quantities σ_n, σ_{nn}, ν_n, ν_{nn} in (3.3) and (3.4) can be chosen in such a way that the elements of the matrix B (B^t) appearing in the rows (columns) passing through those vanishing diagonal elements are also equal to zero

$$\sigma_n = J_n^{mm}, \nu_n = J_n^{kk}, \sigma_{nn} = J_{nn}^{mm}, \nu_{nn} = J_{nn}^{kk} (n = 1,2,\ldots)$$

It follows that the Lagrange multipliers (2.8) are defined completely.

We fix an integer N and set

$$\begin{aligned} \mathbf{w}_6 &= (w_1,\ldots,w_6)^t, \mathbf{w}_N = (w_1,\ldots,w_{6+N})^T \\ \mathbf{w}_* &= (w_{6+N+1}, w_{6+N+2},\ldots)^T \end{aligned}$$

B_N is the $(6 \times N)$-matrix formed by the first N columns of B, B_* is the matrix obtained from B by crossing out the first N columns, and C_N, C_* are the following parts of the matrix C, which are also diagonal matrices

$$\begin{aligned} C_N &= \left\| \omega^{-2} c_{nn} + 2(J_{nn}^{mm} - J_{nn}^{kk} - J_{nn}^{ll}/2) \right\|, n = 1,\ldots,N \\ C_* &= \left\| \omega^{-2} c_{nn} + 2(J_{nn}^{mm} - J_{nn}^{kk} - J_{nn}^{ll}/2) \right\|, n = N+1, N+2,\ldots \end{aligned}$$

The matrix H_1 has a block representation of the general form

$$H_1 = \begin{Vmatrix} A_1 & B_N & B_* \\ B_N^t & C_N & \Theta \\ B_*^t & \Theta & C_* \end{Vmatrix}$$

where Θ is the $(N \times \infty)$ null matrix.

If the diagonal elements of C_* are positive and bounded away from zero, then, extracting a perfect square, one can write

$$\begin{aligned} \delta^2 V_1(0) &= \delta\mathbf{w}_N^t \begin{Vmatrix} (A_1 - B_* C_*^{-1} B_*^t) & B_N \\ B_N^t & C_N \end{Vmatrix} \delta\mathbf{w}_N + \\ &+ (\delta\mathbf{w}_*^t + \delta\mathbf{w}_6^t B_* C_*^{-1}) C_* (\delta\mathbf{w}_* + C_*^{-1} B_*^t \delta\mathbf{w}_6) \end{aligned} \tag{3.5}$$

The conditions of the theorem [4] will be satisfied if the quadratic form $\delta\mathbf{w}_N$ of a finite number of variables in (3.5) is positive-definite on the linear manifolds $\delta U_i = 0$ $(i = 1, 2, 3)$.

Using the well-known method (see, for example, [8]), we arrive at the equivalent problem of studying the conditions for the quadratic form with the real $[(3+N)\times(3+N)]$-matrix

$$H = \begin{Vmatrix} C_N & x_0^t y_0^t & z_0^t \\ x_0 & & \\ y_0 & A & \\ z_0 & & \end{Vmatrix} \tag{3.6}$$

$$x_0 = 2(J_1^{km} + 2q_1^0 J_{11}^{km}, \ldots, J_N^{km} + 2q_N^0 J_{NN}^{km})$$

$$y_0 = \sqrt{3}(J_1^{lm} + 2q_1^0 J_{11}^{lm}, \ldots, J_N^{lm} + 2q_N^0 J_{NN}^{lm})$$

$$z_0 = (J_1^{kl} + 2q_1^0 J_{11}^{kl}, \ldots, J_N^{kl} + 2q_N^0 J_{NN}^{kl})$$

to be positive-definite.

If the rows of infinite length are denoted by

$$x_* = 2(J_p^{km} + 2q_p^0 J_{pp}^{km}, J_{p+1}^{km} + 2q_{p+1}^0 J_{(p+1)(p+1)}^{km}, \ldots)$$

$$y_* = \sqrt{3}(J_p^{lm} + 2q_p^0 J_{pp}^{lm}, J_{p+1}^{lm} + 2q_{p+1}^0 J_{(p+1)(p+1)}^{lm}, \ldots)$$

$$z_* = (J_p^{kl} + 2q_p^0 J_{pp}^{kl}, J_{p+1}^{kl} + 2q_{p+1}^0 J_{(p+1)(p+1)}^{kl}, \ldots), p = N+1$$

and the energy scalar product is introduced by the formulae

$$(x_*, x_*) \equiv x_* C_*^{-1} x_*^t, (x_*, y_*) \equiv x_* C_*^{-1} y_*^T \tag{3.7}$$

then

$$A = \begin{Vmatrix} J_0^{kk} - J_0^{mm} - (x_*, x_*) & -(x_*, y_*) & -(x_*, z_*) \\ -(x_*, y_*) & J_0^{ll} - J_0^{mm} - (y_*, y_*) & -(y_*, z_*) \\ -(x_*, z_*) & -(y_*, z_*) & J_0^{kk} - J_0^{ll} - (z_*, z_*) \end{Vmatrix}$$

In order that H be a positive-definite matrix it is necessary that all the diagonal elements of H, including the diagonal elements of A, be strictly positive.

Next, we choose $\alpha \in (0, 1)$ in such a way that

$$J_0^{kk} - J_0^{ll} = \alpha(J_0^{kk} - J_0^{mm}), J_0^{ll} - J_0^{mm} = (1-\alpha)(J_0^{kk} - J_0^{mm})$$

and introduce the following row vectors of infinite length formed by the rows x_0, x_*; y_0, y_*; z_0, z_*

$$x \equiv (x_0 | x_*), y \equiv (1-\alpha)^{-\frac{1}{2}}(y_0 | y_*), z \equiv \alpha^{-\frac{1}{2}}(z_0 | z_*)$$

We define the energy scalar product of vectors

$$(x, x) \equiv x C^{-1} x^T, (x, y) \equiv x C^{-1} y^T, \ldots \tag{3.8}$$

The roots of the quadratic equation

$$(s - (y, y))(s - (x, x)) - (x, y)^2 = 0 \tag{3.9}$$

arising in the study of the sign of the global diagonal minor of order $N+2$ of H will be denoted by

$$f_1=\frac{1}{2}((x,x)+(y,y))-((\frac{1}{2}((x,x)-(y,y)))^2+(x,y)^2)^{\frac{1}{2}}$$

$$f_2=\frac{1}{2}((x,x)+(y,y))+((\frac{1}{2}((x,x)-(y,y)))^2+(x,y)^2)^{\frac{1}{2}}$$

It is seen that $0<f_1<f_2$. Let a_1, a_2, a_3 be the roots of the cubic equation

$$s^3-s^2p+sg-R=0 \tag{3.10}$$

with

$$p=(x,x)+(y,y)+(z,z),\ g=(y,y)(z,z)-(y,z)^2+$$
$$+(x,x)(y,y)-(x,y)^2+(x,x)(z,z)-(x,z)^2,$$
$$R=\det\begin{Vmatrix}(x,x) & (x,y) & (x,z)\\ (x,y) & (y,y) & (y,z)\\ (x,z) & (y,z) & (z,z)\end{Vmatrix}$$

which arises when the determinant of H is evaluated.

If Eq. (3.10) has one real root, or one simple real root, or one real root 3, then this root is denoted by a_3. In the case of three distinct real roots we assume that $a_1<a_2<a_3$ [9].

The necessary and sufficient conditions for H to be positive definite, which, by Rumyantsev's theorem, are also sufficient for the equilibrium states (2.19), (2.18) to be stable with respect to some of the variables, are given in the following assertion.

Assertion 3. Subject to assumptions 1–5, in order for the positions of relative equilibrium (2.18), (2.19) to be stable with respect to some of the variables Ω_1, Ω_2, Ω_3, q^*, γ_i, β_i, $q_1, \ldots, q_N$, where N is an arbitrary positive number, it is sufficient that the conditions

$$J_0^k>J_0^l>J_0^m \tag{3.11}$$

$$\omega^{-2}c_{nn}+2(J_{nn}^{mm}-J_{nn}^{kk}-J_{nn}^{ll}/2)>\varepsilon,\ n=1,2,\ldots \tag{3.12}$$

$$J_0^k-J_0^m>f_2 \tag{3.13}$$

be satisfied for some $\varepsilon>0$.

If Eq. (3.10) has three distinct real roots, then conditions (3.13) must be supplemented by

$$a_1<J_0^k-J_0^m<a_2 \vee J_0^k-J_0^m>a_3 \tag{3.14}$$

Otherwise the conditions must be supplemented by

$$J_0^k-J^m>a_3,\ J_0^k-J_0^m\neq a_1,a_2 \tag{3.15}$$

The following less cumbersome, but cruder result also holds.

Assertion 4. Under the assumptions made, in order for the positions of relative equilibrium (2.18), (2.19) to be stable with respect to some of the variables Ω_i, $\mathbf{q}^*$, $\mathbf{w}_N$, it is sufficient that H be a matrix with strict diagonal predominance $(i=1, 2, 3)$ [10].

One can treat (3.10) as the condition for a firm of elastic satellite in a position of relative equilibrium to be stable (cf. [1]). Condition (3.12) is analogous to the corresponding conditions in [2, 3]. In the general case the need for a guaranteed difference between the main moments of inertia defined by (3.13) and (3.14) was indicated in [2, 4].

The main difficulty one encounters when searching for the positions of relative equilibrium of the satellite is to solve Eqs (2.18). Approximate solutions can be obtained by using the linear approximation of these equations with respect to q_n^0 and applying the reduction method [11].

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